ARQUIVO 1

The Radial Basis Function Method Applied to Financial Engineering

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Abstract

This paper shows the application of radial basis functions, RBF, to the numerical solution of the classical Black-Scholes, BS, equation, the most important financial engineering tool. The numerical solutions included both the classical and the diffusional form of the BS equation. Thin-Plate Spline (TPS) and Cubic radial basis functions were analyzed in detail as to their ability to allow accurate solutions to the BS equation. A study of feasible numerical solution values, that is, limiting applicable values of stock value mesh size, time step and integration schemes are presented. As a conclusion, it can be affirmed that numerical solutions of BS equation by means of RBF are effective; furthermore, recommendations are presented for solving the equation with a desired accuracy level. The analyses also show that, when RBF techniques are used, there is no significant difference between solutions arising from diffusional and classical BS forms.

Key-words. financial engineering, Black and Scholes, radial basis functions, diffusional method, numerical methods.

O Método de Funções de Base Radial Aplicado à Engenharia Financeira

Resumo

Este artigo mostra a aplicação de funções de base radial, RBF, à solução numérica da equação de Black-Scholes, BS, clássica, a ferramenta mais importante da engenharia financeira. As soluções numéricas incluíram a forma clássica e difusional da equação de BS. As funções radiais de base dos tipos Thin-Plate Spline (TPS) e cúbica foram analisadas em detalhe no que concerne às suas capacidades de permitir soluções acuradas da equação de BS. Apresentam-se os valores limites do tamanho de malha do valor da opção, do passo de tempo e dos esquemas da integração, que podem ser usados para obter soluções numéricas viáveis. Como conclusão, pode-se afirmar que a solução numérica da equação de BS por meio do RBF é eficaz; apresenta-se uma série das recomendações para resolver a equação num nível desejado de exatidão. Além disso, as análises mostram que não há nenhuma diferença significativa entre as soluções numéricas obtidas por meio da forma difusional e da clássica.

Palavras-chave. engenharia financeira, Black e Scholes, funções de base radial, método difusional, métodos numéricos.

Introduction

The most important models of financial engineering are based on Black-Scholes, BS, equation, and are used to predict the outcome of financial options and derivative securities and, thus, help in decision-making processes (COX & RUBINSTEIN, 1985; LEENTVAAR, 2006). Derivative prices are non-linear functions of the underlying risk factors, and the dynamics of the underlying asset follows a stochastic process (MUCHMORE, 2005; CRETIEN, 2006). The BS equation provides insight into the valuation of debt relative to equity (HULL, 1989; SIEGEL et al., 1992). Black-Scholes basic equation is a linear parabolic hyperbolic equation, with stochastic variables and parameters. Improvements on the original model led to a set of non-linear partial differential equations essentially equivalent to the engineering convection-diffusion equation. WHALLEY & WILMOTT (1993), for instance, proposed the following modified form of Black-Scholes equations:

$$\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV - \frac{\sigma^2 S^4 \Gamma^2}{H_0}\left(k_1 + (k_2 + k_3 S)\frac{H_0^{1/2}}{S}\right) = 0 \qquad (1)$$

where V, ?, s, S, r, G, H_0 stand, respectively, for option value, time, volatility, asset (underlying security) price (a stochastic variable), interest rate, option's gamma and a measure of the expected risk of a portfolio; k_1, k_2 and k_3 are cost parameters. It should be emphasized that the two last terms of the equation constitute a non-linear source term. This paper is concerned with the basic formulation of the BS equation, which excludes the last term in equation (1).

Only for some simple assets and simple stochastic environments do we have a closed form analytical solution for the BS equation. In all other cases we have to rely on numerical methods to compute an approximate solution (BARUCCI et al., 1996; GARCIA-OLIVARES, 2003; ZHANG, 2005). It is very difficult to generate stable and accurate solutions to Black-Scholes equations due to the discontinuity of the payoff function around the exercise price (BOZTOSUN & KOC, 2003; CONT & VOLTCHKOVA, 2005).

Many numerical methods have been proposed to model the interaction between advective and diffusive processes. These methods include finite difference, finite element and boundary element methods which are derived from local interpolation schemes and require a mesh to support the application. Finite difference and finite element solutions of the advection–diffusion equation present numerical problems of oscillations and damping (MURPHY & PRENTER, 1985; LEE et al., 1987; ZIENKIEWICZ & TAYLOR, 1991; HOFFMAN, 1992; WILMOTT et al., 1995; WILMOTT, 1998; TOMAS III et al., 2001; BOZTOSUN & CHARAFI, 2002 AMSTER et al., 2003. Zhang (2005) attempted an adaptive finite element procedure to solve the problem of Pricing American options; despite his mathematical reasoning, he did not present any benchmark-based or practical evidence for his claim.

The mesh generation problem over irregularly shaped domains is often in excess of 70% of the total computational cost (BROWN et al., 2005). Thus, meshless methods such as those using Radial Basis Functions have therefore recently attracted much attention from the engineering community and RBFs have become an important tool in scientific computing (BOZTOSUN & CHARAFI, 2002). RBF methods have shown the potential to be a universal grid free method for the numerical solution of partial differential equations (SARRA, 2004). However, when the number of interpolation points used is large and they are densely distributed the resulting solution matrix becomes ill-conditioned. In practice, direct methods for solving the systems resulting from RBFs are inappropriate for problems with more than 2000 interpolation points Other highly accurate spatial discretization schemes such as pseudospectral methods do not have the inherent flexibility of the RBF methods and adaptation and complex geometries are more difficult to deal with (SARRA, 2004).

The diffusional method was proposed by Fortes (1997) and applied to the solutions of several benchmark problems (FORTES & FERREIRA, 1998 and 1999) and is based on transforming the original hyperbolic parabolic partial differential equation into a parabolic partial differential equation; further application of Galerkin's formulation leads to a variational form, readily amenable to computer implementation. Thus, no use is made of ad-hoc Petrov-Galerkin schemes. The method is simple to apply and was shown to perform much better when solving benchmark and practical convection-diffusion problems than the commonly employed finite difference techniques (implicit finite-difference methods including Crank-Nicolson, Douglas schemes, ADI and Hopscotch methods; see HOFFMAN, 1992, for limitations on these methods).

Koc et al(2003) presented a first work on the application of RBF to the solution of BS equation. However, their paper is very succinct and, despite showing some excellent results, does not point to any methodology for obtaining accurate and converging RBF solutions to the BS equation. The objective of this article is to analyze, in a deeper context, the potentiality of the RBF method to solve financial engineering problems. More specifically this paper aims at:

- Presenting Cubic and TPS RBF modeling techniques for solving and optimizing the numerical solution of the BS equation;

- Comparing the effectiveness of the diffusional against the classical form of the BS equation, when both employ the RBF method.

- Evaluating the benefits and limitations of the RBF method with respect to the numerical parameters such as mesh size, time step and integration method.

Methodology

The diffusional and non-diffusional forms of Black-Scholes Equation

The **classical** form of the basic Black-Scholes equation, BS, is (WILMOTT, 1998):

$$\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \qquad (2)$$

In this work, the numerical calculations involved solving the BS equation with a call option with the following payoff function, that is, the value of the call option at expiry (? =T), in a neutral-risk world:

$$\text{Payoff}(S, T) = \max(S\text{-}E, 0) \tag{3}$$

where E is the option exercise or strike value, that is, its value at ? = T. The respective boundary conditions are:

$$V(0, \tau) = 0 \;\; ; \;\; V(\infty,\tau) = S\text{-}Ee^{-r(T-\tau)} \tag{4}$$

One should note that this is not an initial value problem, since the payoff function is given at t = T. In order to make it an initial boundary value problem let us make t = T – ?, so that the above equation becomes

$$\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV = 0 \tag{5}$$

In order to put this last equation into the **diffusional** form, that is, a form that eliminates the convective term, use is made of the identity

$$-\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} = A\frac{\partial}{\partial S}\left(B\frac{\partial V}{\partial S}\right) \tag{6}$$

By comparing the right hand side of equation (20) with its left hand side, after algebraic manipulations, one arrives at:

$$B = \frac{B_0}{(S_0)^{\frac{2r}{\sigma^2}}}(S)^{\frac{2r}{\sigma^2}} \tag{7}$$

and

$$A = -\frac{1}{2}\frac{\sigma^2 S^2}{C_0 S^{\frac{2r}{\sigma^2}}} \tag{8}$$

By substituting the values of A and B into equations (6 and 5), one obtains the **diffusional** form of Black-Scholes equation:

$$\frac{2S^{\frac{2r}{\sigma^2}-2}}{\sigma^2}\frac{\partial V}{\partial t} - \frac{\partial}{\partial S}\left(S^{\frac{2r}{\sigma^2}}\frac{\partial V}{\partial S}\right) + \frac{2S^{\frac{2r}{\sigma^2}-2}rV}{\sigma^2} = 0 \tag{9}$$

The initial and boundary conditions are:

$$V(S,0) = \text{Payoff}(S,0) = \max(S\text{-}E, 0); \quad V(0, t) = 0\; ; V(\infty,t) = S\text{-}Ee^{-r\tau} \tag{10}$$

Radial basis functions applied to the original Black-Scholes equation

The idea behind the RBF method is to use linear translate combinations of a basis function of one variable, expanded about given scattered 'data centers' to approximate an unknown function by

$$V(S,t) = \sum_{i=1}^{N}\lambda_i(t)\,\phi(r_i) = \sum_{i=1}^{N}\lambda_i\,\phi(\|S - S_i\|) \tag{11}$$

where $r_j = \|S - S_i\|$ is the Euclidean norm and λ_i are the coefficients to be determined. Usual radial basis functions are defined by (KOC et al., 2003):

$$\text{Thin-Plate Spline, TPS} \quad : \phi(r_i) = r_i^4 \log(r_i) \tag{12}$$

$$\text{Multiquadrics, MQ} \quad : \phi(r_i) = \sqrt{c^2 + r_i^2} \tag{13}$$

$$\text{Cubic} \quad : \phi(r_i) = r_i^3 \tag{14}$$

$$\text{Gaussian} \quad : \phi(r_i) = e^{-c^2 r_i^2} \tag{15}$$

In this work, only cubic and TPS RBF will be used, due to their simplicity and proven accuracy for other types of problems and the difficulty associated to choosing good values for the shape parameter c, which depends on the problem type (BOZTOSUN and CHARAFI, 2002).

The original Black-Scholes equation shown above in equation (2) can be discretized using the ?-weighted method:

$$\frac{\partial V(S,t)}{\partial t} = f(S,t) \approx (1-\theta)\cdot f(S_t,t) + \theta \cdot f(S_{t+\Delta t}, t+\Delta t) \qquad \text{for } 0 \le \theta \le 1 \tag{16}$$

So, equation (2) becomes:

$$\begin{aligned} &V(S,t) - V(S,t+\Delta t) + \Delta t(1-\theta)\cdot\left[\frac{1}{2}\sigma^2 S^2 \nabla^2 V(S,t) + rS\nabla V(S,t) - rV(S,t)\right] + \\ &+ \Delta t\theta \cdot \left[\frac{1}{2}\sigma^2 S^2 \nabla^2 V(S,t+\Delta t) + rS\nabla V(S,t+\Delta t) - rV(S,t+\Delta t)\right] = 0 \end{aligned} \tag{17}$$

Or

$$\begin{aligned} &V(S,t^n)\cdot\left[1 + \Delta t(1-\theta)\cdot\left(\frac{1}{2}\sigma^2 S^2 \nabla^2 + rS\nabla - r\right)\right] + \\ &+ V(S,t^n+\Delta t)\cdot\left[-1 + \Delta t\theta\cdot\left(\frac{1}{2}\sigma^2 S^2 \nabla^2 + rS\nabla - r\right)\right] = 0 \end{aligned} \tag{18}$$

Defining and, then the previous equation can be written in the form:

$$\left[1 - \alpha\cdot\left(\frac{1}{2}\sigma^2 \mathbf{S}^2 \nabla^2 + r\mathbf{S}\nabla - r\right)\right]\cdot V^{n+1} = \left[1 + \beta\cdot\left(\frac{1}{2}\sigma^2 \mathbf{S}^2 \nabla^2 + r\mathbf{S}\nabla - r\right)\right]\cdot V^n \tag{19}$$

where $\alpha = \theta\Delta t$, $\beta = (1-\theta)\Delta t$, $\nabla = \frac{\partial}{\partial S}$ and $\nabla^2 = \frac{\partial^2}{\partial S^2}$. And, now, by defining two new operators, H_+ and H:

$$H_+ = 1 - \alpha\cdot\left(\frac{1}{2}\sigma^2 \mathbf{S}^2 \nabla^2 + r\mathbf{S}\nabla - r\right), \quad H_- = 1 + \beta\cdot\left(\frac{1}{2}\sigma^2 \mathbf{S}^2 \nabla^2 + r\mathbf{S}\nabla - r\right) \tag{20}$$

equation (14) becomes:

$$\sum_{j=1}^{N} \lambda_j^{n+1} H_+ \phi(\mathbf{S}_{ij}) = \sum_{j=1}^{N} \lambda_j^n H_- \phi(\mathbf{S}_{ij}) \qquad \text{for } i = 1 \ldots N \tag{21}$$

Equation (22) generates a system of linear equations, which can be solved to obtain the unknowns, λ_i^{n+1}, from the known values of λ_i^n at a previous time step. Then they are transformed to the V(**S**, t) by equation (14).

Radial basis functions applied to the diffusional form of Black-Scholes equation

Algebraic manipulations of equation (9), following the ?-weighted method, lead to:

$$\left[\frac{2S^{\frac{2r}{\sigma^2}-2}}{\sigma^2}-\alpha\left(\nabla S^{\frac{2r}{\sigma^2}}\nabla-\frac{2S^{\frac{2r}{\sigma^2}-2}}{\sigma^2}r\right)\right]V^{n+1}=\left[\frac{2S^{\frac{2r}{\sigma^2}-2}}{\sigma^2}+\beta\left(\nabla S^{\frac{2r}{\sigma^2}}\nabla-\frac{2S^{\frac{2r}{\sigma^2}-2}}{\sigma^2}r\right)\right]V^{n} \quad (22)$$

The operators H_+ and H_- are then defined by:

$$H_+=\left[\frac{2S^{\frac{2r}{\sigma^2}-2}}{\sigma^2}-\alpha\left(\nabla S^{\frac{2r}{\sigma^2}}\nabla-\frac{2S^{\frac{2r}{\sigma^2}-2}}{\sigma^2}r\right)\right] \quad , \quad H_-=\left[\frac{2S^{\frac{2r}{\sigma^2}-2}}{\sigma^2}+\beta\left(\nabla S^{\frac{2r}{\sigma^2}}\nabla-\frac{2S^{\frac{2r}{\sigma^2}-2}}{\sigma^2}r\right)\right] \quad (23)$$

Finally, the new operators H_+ and H_-, are applied in equation (21), generating a system of RBF linear equation given by and equivalent to equation (21).

Analytical solution

The analytical solution for the present value of a call option is given by:

$$V(S,T-t)=S\cdot N(d_1)-E\cdot N(d_2)\cdot e^{r(T-t)} \quad (24)$$

where

$$d_1(S,T-t)=\frac{\ln\left(\frac{S}{E}\right)+\left(r+\frac{1}{2}\sigma^2\right)\cdot(T-t)}{\sigma\cdot\sqrt{T-t}} \quad (25)$$

$$d_2(S,T-t)=d_1(S,T-t)-\sigma\cdot\sqrt{T-t} \quad (26)$$

and N is the cumulative normal probability density function,

$$N(x)=(2\pi)^{-\frac{1}{2}}\int_0^x e^{-\frac{1}{2}u^2}du+\frac{1}{2} \quad (27)$$

The instantaneous Payoff function, M, is given by

$$M(S,T-t))==\begin{cases}S-Ee^{-r(T-t)}, \text{if } S-Ee^{-r(T-t)}\geq 0\\ 0, \text{otherwise}\end{cases} \quad (28)$$

Results and Discussion

The results to be shown were obtained via Mathcad, a symbolic mathematical programming language and solver. Option value or price or premium is defined as the call option value that an option buyer pays the seller. The fixed data used in the simulation studies were:

- Exercise value = E = 50
- Volatility = s = 20%
- Riskless interest rate = r = 5%
- Expiry time = T = 1
- Present exact analytical call option value = 5.225

The total number of stock price meshes is N and the mesh size, ΔS, is defined by $\Delta S = S/N$. The total number of time steps is Nt, while the time step, Δt, is defined by $\Delta t = T/Nt$.

In this work, numerical option value relative errors refer to option prediction values at the strike (exercise) price value (S = E = 50) and are defined as:

$$\varepsilon = \text{Relative error of numerical option price (\%)} = \frac{\text{Numcrical option valuc-Analytical solution valuc}}{\text{Analytical solution value}} \times 100\%$$

One of the boundary conditions, typical in BS problems, requires specifying V(S, t) at S = ; practical numerical solutions require that this value should be reduced and, the larger the allowable reduction, the better the efficiency of the numerical solution, due to decreased equation matrix size; thus, the practical maximum simulated value for S was called S_{max}.
The accuracy of *finite difference* solutions of BS equations can be heavily improved if the diffusional method substitutes the classical approach (FORTES et al, 2005). However, in this work, when RBF are considered, the diffusional and the classical form of Black-Scholes equation led to the same results. Thus, this fact will not be shown in the figures to come.
Furthermore, as shown in Figure 1, excellent solutions can be obtained by any of the cubic or TPS radial basis functions. Figure 1 was obtained with Nt =100 time steps, N =112 meshes and ΔS = 0.714, with an upper value of S equal to 80. With these parameters, the cubic RBF led to an option price relative error of 0.00039% at the exercise option value (E=50), while in the case of the TPS RBF, the relative error was 0.019%.

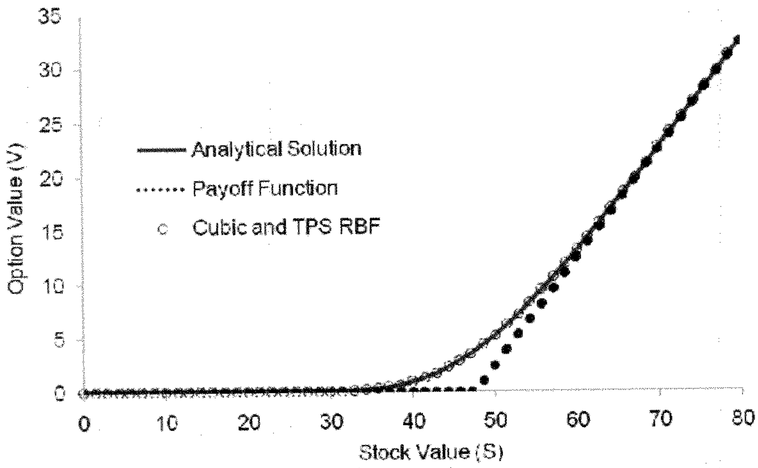

Figure 1. Cubic and TPS RBF simulated values of a call option, V, compared against the analytical solution and payoff function values; E = 50, T = 1, s = 20%, r = 5%,

Figure 2 shows the effect of the integration scheme; the ?-variation was performed based on Nt = 100, N = 112, S_{max} = 80 for cubic RBF and Nt = 300, N = 200 and $S_{máx}$ = 100, in the case of TPS RBF. As can be noted, it is advisable to use implicit schemes with $\theta \geq 0.5$, as a general rule; smaller θ values lead to divergence. Although the choice of θ can affect the accuracy of numerical solutions, Figure 2 allows affirming that a good choice is to use a ? value slightly larger than the one that leads to divergence. Even higher values of θ still provide highly accurate solutions. One can notice that Cubic RBF is more efficient than TPS RBF, since cubic RBF require less time steps, a smaller S_{max} and less meshes for approximately similar error levels. Changes in ΔS, Δt and S_{max}, that is, the other simulation parameters, did not affect the just mentioned conclusions.

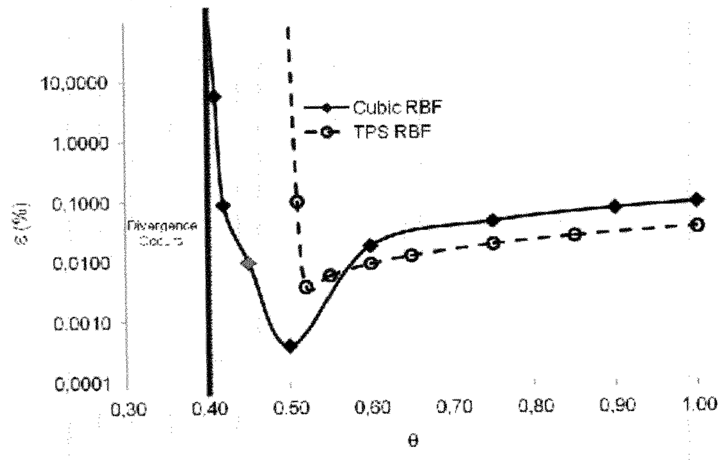

Figure 2. Relative errors of cubic and TPS RBF as affected by the integration θ-value. E = 50, T = 1, σ = 20%, r = 5%; Nt = 100, N = 112, $S_{máx}$ = 80 for cubic RBF and Nt = 300, N = 200 and $S_{máx}$ = 100, in the case of TPS RBF.

The effect of the number of time steps on the accuracy of the numerical solutions can be visualized in Figure 3. As can be noticed, Cubic RBF performs better than TPS RBF. TPS error decreases as the time step (= T/Nt) decreases or the number of time steps increase; however, if the number of time steps gets lower than a limiting value (300), divergence occurs.. On the other hand, cubic RBF lead to smaller errors and requires less times steps and presents a point of minimum error. Thus, as a general rule, TPS RBF should be used with larger number of time steps or smaller time steps, while the optimal number of time steps for cubic RBF can be obtained by starting with a large number of times steps and reducing it by increasing the time step and observing the solution behavior.

Figure 4 shows the numerical solution behavior of both RB functions, as affected by the number of grid points or, inversely, the stock value mesh size. As can be noticed, TPS RBF diverges when the number of grid points exceeds 200; interestingly enough, at this grid point number, it reaches the smallest relative error. Cubic RBF behave excellently well, with very small relative option value errors. Proper choice of Cubic RBF involves checking the solution consistency at a larger number of grid points. Thus, Figure 4 allows recommending the use of the maximum acceptable value of grid points, as a general rule since it leads to accurate values. In the investigated literature, there is no known technique for finding the minimum error point for the RBF method.

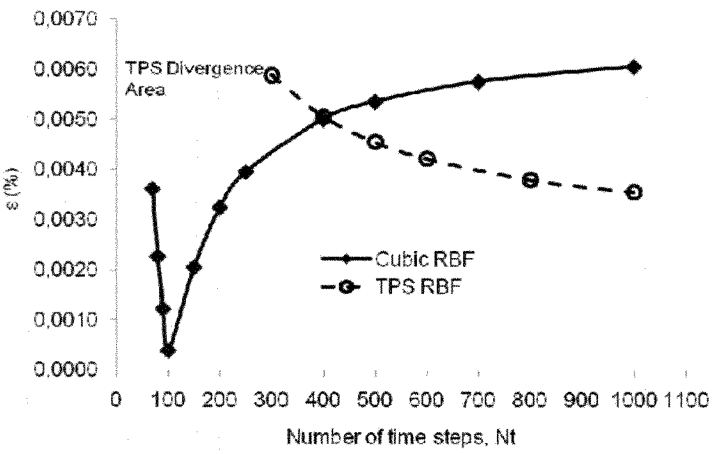

Figure 3. Relative errors of TPS and Cubic RBF call option values at the exercise option value (%) as affected by the number of time steps, Nt. E = 50, T = 1, σ = 20%, r = 5%; N = 112, $S_{máx}$ = 80 for cubic RBF and N = 200 and $S_{máx}$ = 100, in the case of TPS RBF.

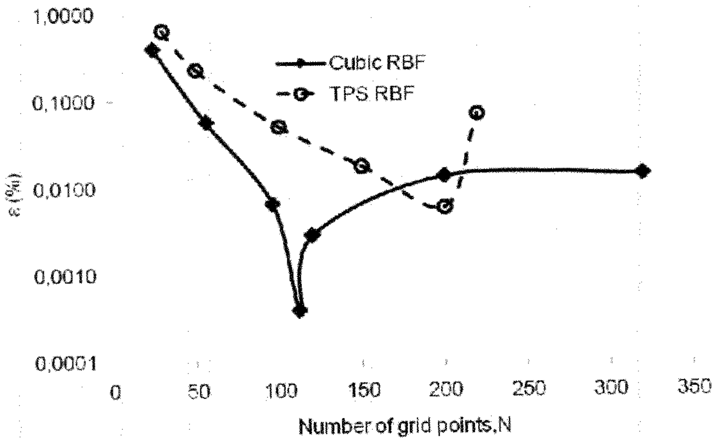

Figure 4. Relative errors of TPS and Cubic RBF call option values at the exercise option value (%) as affected by the number of grid points, N. E = 50, T = 1, σ = 20%, r = 5%; Nt = 100, $S_{máx}$ = 80 for cubic RBF and Nt = 300 and $S_{máx}$ = 100, in the case of TPS RBF.

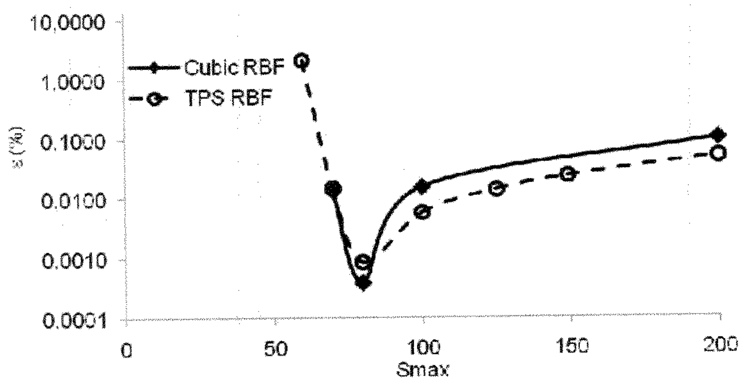

Figure 5 shows that there is a value of S_{max} that minimizes the relative error of the numerical solutions through any of the RB functions; however low values can increase the error or lead to divergence. So, the numerical results recommend using S_{max} ¤ 2E.

Figure 5. Relative error at exercise option value (%) vs. Maximum Stock Value, S_{max}

Conclusions

A methodology based on the Radial Basis Functions is presented and shown to be very effective and reliable to solve the most important finance engineering equation, that is, Black-Scholes equation.

Based on the research results, the following conclusions can be stated:

1 - In the context of the RBF method, the diffusional method and the classical form of the BS equation lead to the same results and can be used interchangeably.

2 - Both TPS and Cubic radial functions furnish accurate results to the BS equation. There is no definitive evidence of sensitive differences between both methods.

3 - Excellent results can be obtained through:

a - Using an integration method with $\theta \geq 0,5$

b - Using a high number of time steps in the case of the TPS RBF and the lowest convergent number for Cubic RBF.

c - Using the minimum convergent value for the stock mesh size for both RBF.

d - Using a maximum value for the stock price approximately equal to two times the exercise price

Thus, the methodology herein presented should be of help to the effective use of the RBF method to the solution of Black and Scholes equation.

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